Lesson 12: Solving Equations

Student Outcomes

- Students are introduced to the formal process of solving an equation: starting from the assumption that the original equation has a solution. Students explain each step as following from the properties of equality. Students identify equations that have the same solution set.

Classwork

Opening Exercise (4 minutes)

Opening Exercise

Answer the following questions.

a. **Why should the equations $x - 1 \quad x + 3 \quad = 17 + x$ and $x - 1 \quad x + 3 \quad = x + 17$ have the same solution set?**

The commutative property.

b. **Why should the equations $x - 1 \quad x + 3 \quad = 17 + x$ and $x + 3 \quad x - 1 \quad = 17 + x$ have the same solution set?**

The commutative property.

c. **Do you think the equations $(x - 1)(x + 3) = 17 + x$ and $x - 1 \quad x + 3 \quad + 500 = 517 + x$ should have the same solution set? Why?**

Yes, 500 was added to both sides.

d. **Do you think the equations $x - 1 \quad x + 3 \quad = 17 + x$ and $3(x - 1)(x + 3) = 51 + 3x$ should have the same solution set? Explain why.**

Yes, both sides were multiplied by 3.

Discussion (4 minutes)

Allow students to attempt to justify their answers for (a) and (b) above. Then summarize with the following:

- We know that the commutative and associative properties hold for all real numbers. We also know that variables are placeholders for real numbers, and the value(s) assigned to a variable that make an equation true is the "solution." If we apply the commutative and associative properties of real numbers to an expression, we obtain an equivalent expression. Therefore, equations created this way (by applying the commutative and associative properties to one or both expressions) consist of expressions equivalent to those in the original equation.
- In other words, if x is a solution to an equation, then it will also be a solution to any new equation we make by applying the commutative and associative properties to the expression in that equation.

Exercise 1 (3 minutes)

Exercise 1

a. **Use the commutative property to write an equation that has the same solution set as**

$$\boldsymbol{x^2 - 3x + 4 = (x+7)(x-12)(5).}$$

$$-3x + x^2 + 4 = (x+7)(5)(x-12)$$

b. **Use the associative property to write an equation that has the same solution set as**

$$\boldsymbol{x^2 - 3x + 4 = (x+7)(x-12)(5).}$$

$$x^2 - 3x \quad + 4 = \quad x + 7 \quad x - 12 \quad (5)$$

c. **Does this reasoning apply to the distributive property as well?**

Yes, it does apply to the distributive property.

Discussion (3 minutes)

- Parts (c) and (d) of the Opening Exercise rely on key properties of equality. What are they?

Call on students to articulate and compare their thoughts as a class discussion. In middle school, these properties are simply referred to as the *if-then moves*. Introduce their formal names to the class, the *additive and multiplicative properties of equality*. Summarize with the following as you write it on the board:

- So, whenever $a = b$ is true, then $a + c = b + c$ will also be true for all real numbers c.
- What if $a = b$ is false?
  - *Then $a + c = b + c$ will also be false.*
- Is it also ok, to subtract a number from both sides of the equation?
  - *Yes, this is the same operation as adding the opposite of that number.*
- Whenever $a = b$ is true, then $ac = bc$ will also be true, and whenever $a = b$ is false, $ac = bc$ will also be false for all non-zero real numbers c.
- So, we have said earlier that applying the distributive, associative, and commutative properties does not change the solution set, and now we see that applying the additive and multiplicative properties of equality also preserves the solution set (does not change it).
- Suppose I see the equation $x + 5 = 2$. (Write the equation on the board.)
- Is it true, then, that $x + 5 - 5 = 2 - 5$? (Write the equation on the board.)

MP.1

Allow students to verbalize their answer and challenge each other if they disagree. If they all say, "yes", prompt students with, "Are you sure?" until one or more students articulate that we would get the false statement: $x = -3$. Then summarize with the following points. Give these points great emphasis, perhaps adding grand gestures or voice inflection to recognize the importance of this moment, as it addresses a major part of **A-REI.A.1**:

- So our idea that adding the same number to both sides gives us another true statement depends on the idea that the first equation has a value of x that makes it true to begin with.

MP.1

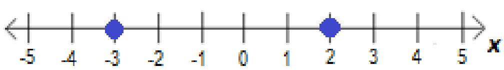
- It is a big assumption that we make when we start to solve equations using properties of equality. We are assuming there is some value for the variable that makes the equation true. **IF** there is, then it makes sense that applying the properties of equality will give another true statement. But we must be cognizant of that big "IF".
- What if there was no value of x to make the equation true? What is the effect of adding a number to both sides of the equation or multiplying both sides by a non-zero number?
  - *There still will be no value of x that makes the equation true. The solution set is still preserved; it will be the empty set.*
- Create another equation that initially seems like a reasonable equation to solve but in fact has no possible solution.

Exercise 2 (7 minutes)

Exercise 2

Consider the equation $x^2 + 1 = 7 - x$.

a. **Verify that this has the solution set $\{-3, 2\}$. Draw this solution set as a graph on the number line. *We will later learn how to show that these happen to be the ONLY solutions to this equation.***



$2^2 + 1 = 7 - 2$ True. $(-3)^2 + 1 = 7 - -3$ True.

b. **Let's add four to both sides of the equation and consider the new equation $x^2 + 5 = 11 - x$. Verify 2 and -3 are still solutions.**

$2^2 + 5 = 11 - 2$ True. $(-3)^2 + 5 = 11 - (-3)$ True. They are still solutions.

c. **Let's now add x to both sides of the equation and consider the new equation $x^2 + 5 + x = 11$. Are 2 and -3 still solutions?**

$2^2 + 5 + 2 = 11$ True. $(-3)^2 + 5 + -3 = 11$ True. 2 and -3 are still solutions.

d. **Let's add -5 to both sides of the equation and consider the new equation $x^2 + x = 6$. Are 2 and -3 still solutions?**

$2^2 + 2 = 6$ True. $(-3)^2 + -3 = 6$ True. 2 and -3 are still solutions.

e. **Let's multiply both sides by $\frac{1}{6}$ to get $\frac{x^2+x}{6} = 1$. Are 2 and -3 still solutions?**

$\frac{2^2+2}{6} = 1$ True. $\frac{(-3)^2+(-3)}{6} = 1$ True. 2 and -3 are still solutions.

f. **Let's go back to part (d) and add $3x^3$ to both sides of the equation and consider the new equation $x^2 + x + 3x^3 = 6 + 3x^3$. Are 2 and -3 still solutions?**

$2^2 + 2 + 3(2)^3 = 6 + 3(2)^3$ $(-3)^2 + -3 + 3(-3)^3 = 6 + 3(-3)^3$

$4 + 2 + 24 = 6 + 24$ True. $9 - 3 - 81 = 6 - 81$ True.

2 and -3 are still solutions.

Discussion (4 minutes)

- In addition to applying the commutative, associative, and distributive properties to equations, according to the exercises above, what else can be done to equations that does not change the solution set?
  - *Adding a number to or subtracting a number from both sides.*
  - *Multiplying or dividing by a non-zero number.*
- What we discussed in Example 1 can be rewritten slightly to reflect what we have just seen: If x is a solution to an equation, it will also be a solution to the new equation formed when the same number is added to (or subtracted from) each side of the original equation or when the two sides of the original equation are multiplied by the same number or divided by the same non-zero number. These are referred to as the Properties of Equality. This now gives us a strategy for finding solution sets.
- Is $x = 5$ an equation? If, so what is its solution set?
  - *Yes, its solution set is* 5.
- This example is so simple that it is hard to wrap your brain around, but it points out that if ever we have an equation that is this simple, we know its solution set.
- We also know the solution sets to some other simple equations, such as

  (a) $w^2 = 64$ (b) $7 + P = 5$ (c) $3\beta = 10$

- Here's the strategy:

  If we are faced with the task of solving an equation, that is, finding the solution set of the equation:

  Use the commutative, associative, distributive properties

  AND

  Use the properties of equality (adding, subtracting, multiplying, dividing by non-zeros)

  to keep rewriting the equation into one whose solution set you easily recognize. (We observed that the solution set will not change under these operations.)

- This usually means rewriting the equation so that all the terms with the variable appear on one side of the equation.

Exercise 3 (5 minutes)

Exercise 3

a. **Solve for r: $\frac{3}{2r} = \frac{1}{4}$**

$r = 6$

b. **Solve for s: $s^2 + 5 = 30$**

$s = 5, s = -5$

c. **Solve for y: $4y - 3 = 5y - 8$**

$y = 5$

Exercise 4 (5 minutes)

- Does it matter which step happens first? Let's see what happens with the following example.

Do a quick count-off or separate the class into quadrants. Give groups their starting points. Have each group designate a presenter, so the whole class can see the results.

MP.3

Exercise 4

Consider the equation $3x + 4 = 8x - 16$. Solve for x using the given starting point.

Group 1	Group 2	Group 3	Group 4
Subtract $3x$ from both sides	*Subtract 4 from both sides*	*Subtract $8x$ from both sides*	*Add 16 to both sides*
$3x + 4 - 3x = 8x - 16 - 3x$ $4 = 5x - 16$ $4 + 16 = 5x - 16 + 16$ $20 = 5x$ $\frac{20}{5} = \frac{5x}{5}$ $4 = x$ $\{4\}$	$3x + 4 - 4 = 8x - 16 - 4$ $3x = 8x - 20$ $3x - 8x = 8x - 20 - 8x$ $-5x = -20$ $\frac{-5x}{-5} = \frac{-20}{-5}$ $x = 4$ $\{4\}$	$3x + 4 - 8x = 8x - 16 - 8x$ $-5x + 4 = -16$ $-5x + 4 - 4 = -16 - 4$ $-5x = -20$ $\frac{-5x}{-5} = \frac{-20}{-5}$ $x = 4$ $\{4\}$	$3x + 4 + 16 = 8x - 16 + 16$ $3x + 20 = 8x$ $3x + 20 - 3x = 8x - 3x$ $20 = 5x$ $\frac{20}{5} = \frac{5x}{5}$ $4 = x$ $\{4\}$

- Therefore, according to this exercise, does it matter which step happens first? Explain why or why not.
  - *No, because the properties of equality produce equivalent expressions, no matter the order in which they happen.*
- How does one know "how much" to add/subtract/multiply/divide? What's the goal of using the properties and how do they allow equations to be solved?

Encourage students to verbalize their strategies to the class and to question each other's reasoning and question the precision of each other's description of their reasoning. From middle school, students recall that the goal is to isolate the variable by making 0s and 1s. Add/subtract numbers to make the zeros, and multiply/divide numbers to make the 1s. The properties say any numbers will work, which is true, but with the 0s and 1s goal in mind, equations can be solved very efficiently.

- The ability to pick the most efficient solution method comes with practice.

Closing Exercise (5 minutes)

Answers will vary. As time permits, share several examples of student responses.

MP.2

Closing Exercise

Consider the equation $3x^2 + x = (x - 2)(x + 5)x$.

a. Use the commutative property to create an equation with the same solution set.

$x + 3x^2 = (x + 5)(x - 2)x$

b. Using the result from (a), use the associative property to create an equation with the same solution set.

$x + 3x^2 \quad = \quad x + 5 \quad x - 2 \quad x$

c. Using the result from (b), use the distributive property to create an equation with the same solution set.

$x + 3x^2 = x^3 + 3x^2 - 10x$

MP.2

d. **Using the result from (c), add a number to both sides of the equation.**

$\mathbf{x + 3x^2 + 5 = x^3 + 3x^3 - 10x + 5}$

e. **Using the result from (d), subtract a number from both sides of the equation.**

$\mathbf{x + 3x^2 + 5 - 3 = x^3 + 3x^2 - 10x + 5 - 3}$

f. **Using the result from (e), multiply both sides of the equation by a number.**

$\mathbf{4\ \ x + 3x^2 + 2\ \ = 4\ \ x^3 + 3x^2 - 10x + 2}$

g. **Using the result from (f), divide both sides of the equation by a number.**

$\mathbf{x + 3x^2 + 2 = x^3 + 3x^2 - 10x + 2}$

h. **What do all seven equations have in common? Justify your answer.**

They will all have the same solution set.

Lesson Summary

If x is a solution to an equation, it will also be a solution to the new equation formed when the same number is added to (or subtracted from) each side of the original equation or when the two sides of the original equation are multiplied by (or divided by) the same non-zero number. These are referred to as the *Properties of Equality*.

If one is faced with the task of solving an equation, that is, finding the solution set of the equation:

Use the *commutative*, *associative*, and *distributive properties*, AND use the *properties of equality* (adding, subtracting, multiplying by non-zeros, dividing by non-zeros) to keep rewriting the equation into one whose solution set you easily recognize. (We believe that the solution set will not change under these operations.)

Exit Ticket (5 minutes)

Name ____ Date____________________

Lesson 12: Solving Equations

Exit Ticket

Determine which of the following equations have the same solution set by recognizing properties, rather than solving.

a. $2x + 3 = 13 - 5x$

b. $6 + 4x = -10x + 26$

c. $6x + 9 = \frac{13}{5} - x$

d. $0.6 + 0.4x = -x + 2.6$

e. $3(2x + 3) = \frac{13}{5} - x$

f. $4x = -10x + 20$

g. $15\ \ 2x + 3\ \ = 13 - 5x$

h. $15\ \ 2x + 3\ \ + 97 = 110 - 5x$

Exit Ticket Sample Solutions

Determine which of the following equations have the same solution set by recognizing properties, rather than solving.

a. $2x + 3 = 13 - 5x$

b. $6 + 4x = -10x + 26$

c. $6x + 9 = \frac{13}{5} - x$

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f. $4x = -10x + 20$

g. $15\ \ 2x + 3\ \ = 13 - 5x$

h. $15\ \ 2x + 3\ \ + 97 = 110 - 5x$

(a), (b), (d), and (f) have the same solution set. (c), (e), (g), and (h) have the same solution set.

Problem Set Sample Solutions

1. **Which of the following equations have the same solution set? Give reasons for your answers that do not depend on solving the equations.**

   I. $x - 5 = 3x + 7$

   II. $3x - 6 = 7x + 8$

   III. $15x - 9 = 6x + 24$

   IV. $6x - 16 = 14x + 12$

   V. $9x + 21 = 3x - 15$

   VI. $-0.05 + \frac{x}{100} = \frac{3x}{100} + 0.07$

   I, V, and VI all have the same solution set; V is the same as I after multiplying both sides by 3 and switching the left side with the right side; VI is the same as 1 after dividing both sides by 100 and using the commutative property to rearrange the terms on the left side of the equation.

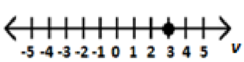
   II and IV have the same solution set. IV is the same as II after multiplying both sides by 2 and subtracting 4 from both sides.

   III does not have the same solution set as any of the others.

Solve the following equations, check your solutions, and then graph the solution sets.

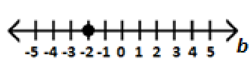
2. $-16 - 6v = -2(8v - 7)$

   3



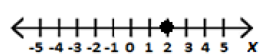
3. $2(6b + 8) = 4 + 6b$

   -2



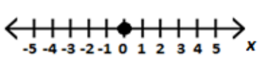
4. $x^2 - 4x + 4 = 0$

   2



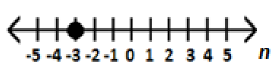
5. $7 - 8x = 7\ \ 1 + 7x$

   0



6. $39 - 8n = -8(3 + 4n) + 3n$

   -3



7. $x - 1\ \ x + 5\ \ = x^2 + 4x - 2$

   no solution

8. $x^2 - 7 = x^2 - 6x - 7$

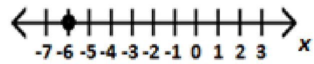
0

9. $-7 - 6a + 5a = 3a - 5a$

7

10. $7 - 2x = 1 - 5x + 2x$

-6



11. $4\ x - 2\ = 8\ x - 3\ - 12$

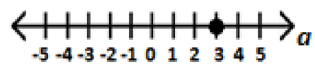
7

12. $-3\ 1 - n\ = -6 - 6n$

$-\frac{1}{3}$

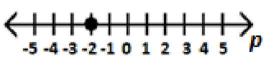
13. $-21 - 8a = -5(a + 6)$

3



14. $-11 - 2p = 6p + 5(p + 3)$

-2

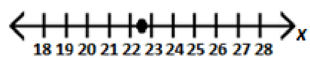


15. $\frac{x}{x+2} = 4$

$-\frac{8}{3}$

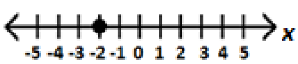
16. $2 + \frac{x}{9} = \frac{x}{3} - 3$

$\frac{45}{2}$



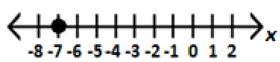
17. $-5\ -5x - 6\ = -22 - x$

-2



18. $\frac{x+4}{3} = \frac{x+2}{5}$

-7



19. $-5\ 2r - 0.3\ + 0.5\ 4r + 3\ = -64$

$\frac{67}{8}$